Winter School at UACEG

Topic: Water Management Optimization Problems

Task for Students # 2:

Transportation Problem – Application in Water Resources Management.

Explanations and Example

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TASK # 2

OPTIMIZATION PROBLEMS.

TRANSPORTATION PROBLEM – APPLICATION IN WATER RESOURCES MANAGEMENT

1. Initial data

A simplified scheme of the water management system (WMS) is presented on Fig. 1.

According to initial data provided in *Terms of Reference (TOR)* the WMS consists of 3 water sources – a pumping station (PS 1) abstracting water from a reservoir, a pumping station (PS 2) abstracting surface water and a pumping station (PS 3) abstracting groundwater. These water sources are named A_1, A_2 и A_3. The water sources have supply capacities of the following volumes of water per day (in thousands m^3): $W_1^A = 40$, $W_2^A = 40$ и $W_3^A = 20$.

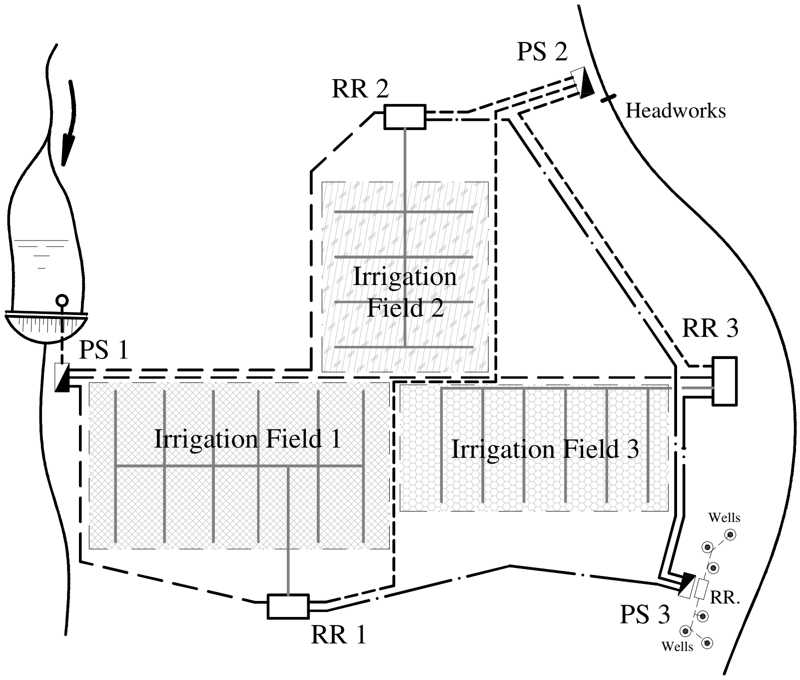

Fig. 1. Schematic View of the Water Sources, Delivery Networks and Water Users

The water users are 3 Irrigation Fields (IFs), together with their Regulating Reservoirs (RRs), which are located at a command elevation above the IFs. Generally, the water users are named B_1, B_2 и B_3.

The water demand per day of these three water users are as follows: $W_1^B = 30$, $W_2^B = 60$, $W_3^B = 30$.

Each water source A_i can supply water to each water user B_j.

The costs $Z_{i,j}$ for water delivery from each water source A_i to each water user B_j are provided in Annex 2 of *TOR*. The costs $Z_{i,j}$ are not constants, nor in linear relation to the volumes of water supplied from water source A_i to water user B_j. The functions “Costs $Z_{i,j}$” versus “Supplied volumes $V_{i,j}$” are shown as graphs on Fig. 2.

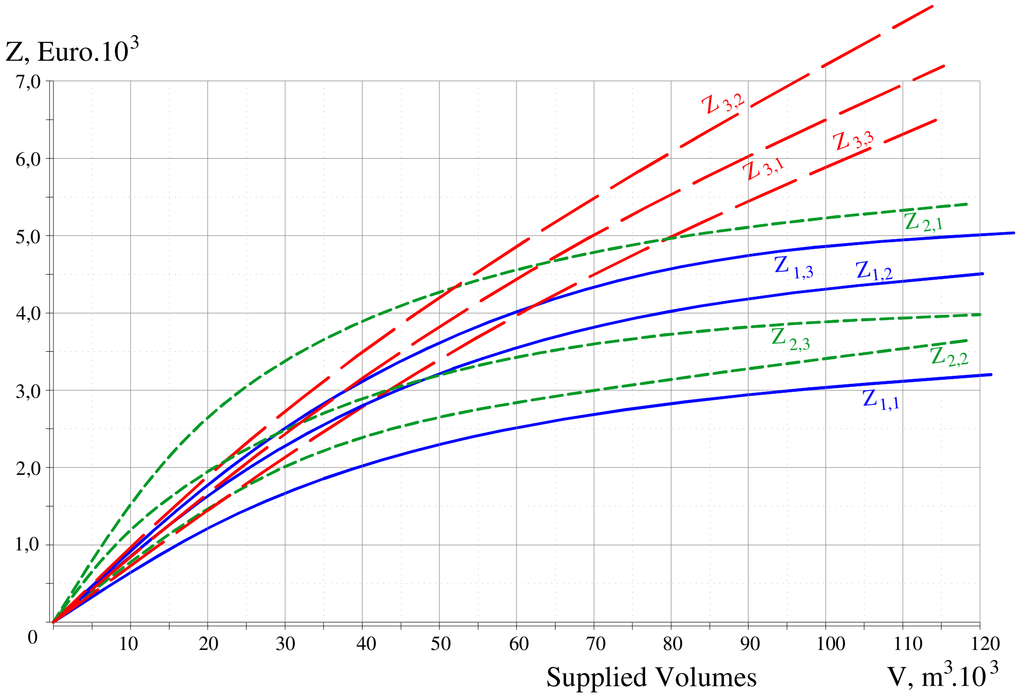

Fig. 2. Water Supply Cost Functions

The minimum cost for supplying water from 3 water sources to 3 water users has to be found, as well as the volumes $V_{i,j}$ supplied from each water source A_i to each water user B_j have to be determined.

2. Problem solution

2.1. Type of Transportation problem

The type of transportation problem is determined by checking if the *total supply* (total volumes available) is equal to the *total demand*:

$$\sum_i W_i^A \overset{\le}{\underset{\ge}{=}} \sum_j W_j^B$$

<u>Check:</u>

Total supply (available volumes in water sources A_i): $\sum_i W_i^A = 40 + 40 + 20 = 100$ m^3.10^3.

Total demand (needed volumes by water users B_j): $\sum_j W_j^B = 30 + 60 + 30 = 120$ m^3.10^3.

It is evident that $\sum_i W_i^A < \sum_j W_j^B$, thus <u>*the problem is not balanced*</u>.

The disbalance is: $\sum_i W_i^A - \sum_j W_j^B = 100 - 120 = -20$ m^3.10^3. (i.e. deficit is 20 m^3.10^3.).

A virtual water source A_4 is introduced (it is called *dummy* water source). The virtual capacity of the dummy water source is $W_4^A = 20$ m^3.10^3, thus the problem becomes balanced.

2.2. Objective Function and Constraints

2.2.1. Objective Function

The Objective Function (OF) is a total cost for supplying water from all water sources A_i to all water users B_j. The classic form of the OF is:

$$Z = V_{11}C_{11} + V_{12}C_{12} + V_{13}C_{13} + V_{21}C_{21} + V_{22}C_{22} + V_{23}C_{23} + V_{31}C_{31} + V_{32}C_{32} + V_{33}C_{33} +$$
$$+ V_{41}C_{41} + V_{42}C_{42} + V_{43}C_{43}, €. \quad (1)$$

or shortly: $Z = \sum_i \sum_j V_{ij} C_{ij}$, as $i = 1 \div 4$; $j = 1 \div 3$,

where $V_{i,j}$ are the volumes of eater supplied from water source A_i to all water user B_j;
$C_{i,j}$ – the unit costs for transporting (supplying) water from water source A_i to all water user B_j.

Since the unit costs $C_{i,j}$ for supplying water from water source A_i to all water user B_j are not constant, but variable in function of the volumes supplied, as it is provided by ***TOR***, a new form of OF is defined:

$$Z = \sum_i \sum_j Z_{ij}, €, \quad (2)$$

where $Z_{ij} = V_{ij}C_{ij}$ is the cost for supplying water from water source A_i to all water user B_j .

2.2.2. Constraints

a) Supply Constraints

These constraints take into account the capacity of water sources.

$$\begin{vmatrix} V_{11} + V_{12} + V_{13} = W_1^A \\ V_{21} + V_{22} + V_{23} = W_2^A \\ V_{31} + V_{32} + V_{33} = W_3^A \\ V_{41} + V_{42} + V_{43} = W_4^A \end{vmatrix} \quad (3)$$

b) Demand Constraints

These constraints take into account the demand of water users.

$$\begin{vmatrix} V_{11} + V_{21} + V_{31} + V_{41} = W_1^B \\ V_{12} + V_{22} + V_{32} + V_{42} = W_2^B \\ V_{13} + V_{23} + V_{33} + V_{43} = W_3^B \end{vmatrix} \quad (4)$$

2.3. Creation of Solution Model in MS Excel

2.3.1. Cost functions

Explanations: The following text in blue and figures with Latin numbers (i, ii, iii, etc.) **should not** be written in the explanatory note. The text and figures are used to describe the procedures and to help solving the task.

The graphs in Annex 2 have to be replaced and presented as equations. Thus, the equations can be used in MS Excel to calculate the cost for a given volume of water supplied.

To find equation of each function $Z_{i,j} - V$ one should make readings for 3 or 4 points from the graphs on Fig. 2. A table is made in MS Excel (see **Fig. i**). It is convenient to read values $Z_{i,j}$ for the same values of the volume V.

COSTS READ FROM THE GRAPH

	Volumes of Water				
Cost	0	20	40	70	100
Z 1,1	0	1,21	2,02	2,69	3,03
Z 1,2	0	1,63	2,8	3,82	4,31
Z 1,3	0	1,77	3,11	4,33	4,86
Z 2,1	0	2,65	3,89	4,78	5,23
Z 2,2	0	1,46	2,39	3	3,41
Z 2,3	0	1,95	2,89	3,6	3,89
Z 3,1	0	1,66	3,15	5	6,5
Z 3,2	0	1,88	3,49	5,48	7,21
Z 3,3	0	1,44	2,78	4,5	5,88
Z 4,1					
Z 4,2					
Z 4,3					

Fig. i

Then, using the MS Excel option Insert→Graph→ X Y Scatter graphs are inserted into the worksheet (see **Fig. ii**).

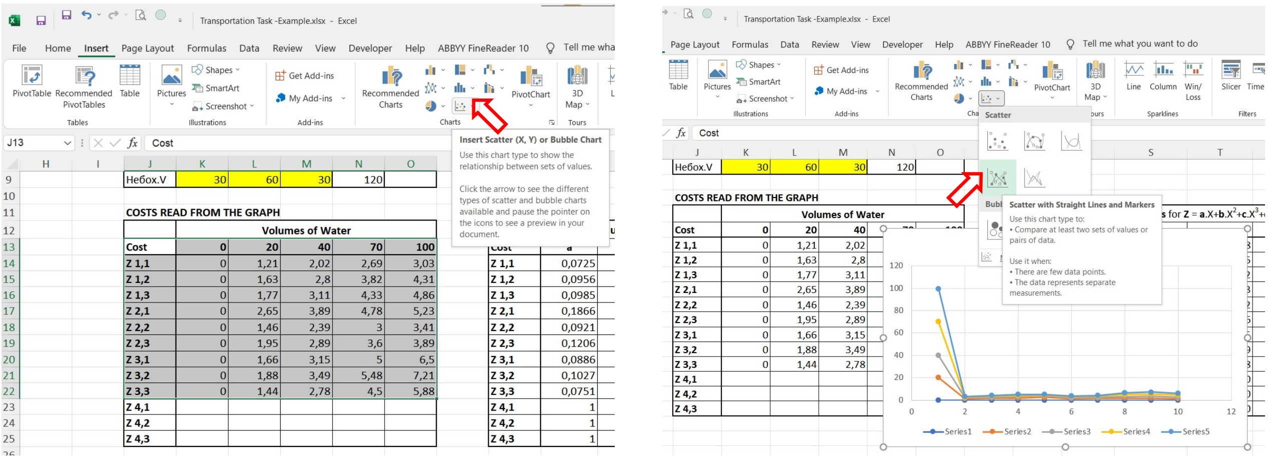

Fig. ii

The initial graph is not the desired one. The row and columns should be switched to obtain needed result (see Fig. iii).

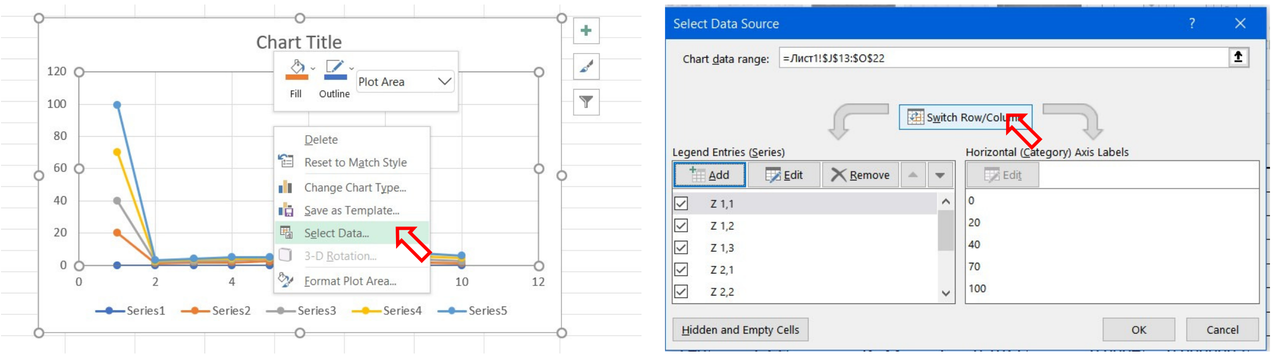

Fig. iii

Then, by clicking on a given function with the right mouse button and selecting the option Add Trendline from the context menu, a trendline is inserted (**Fig. iv**).

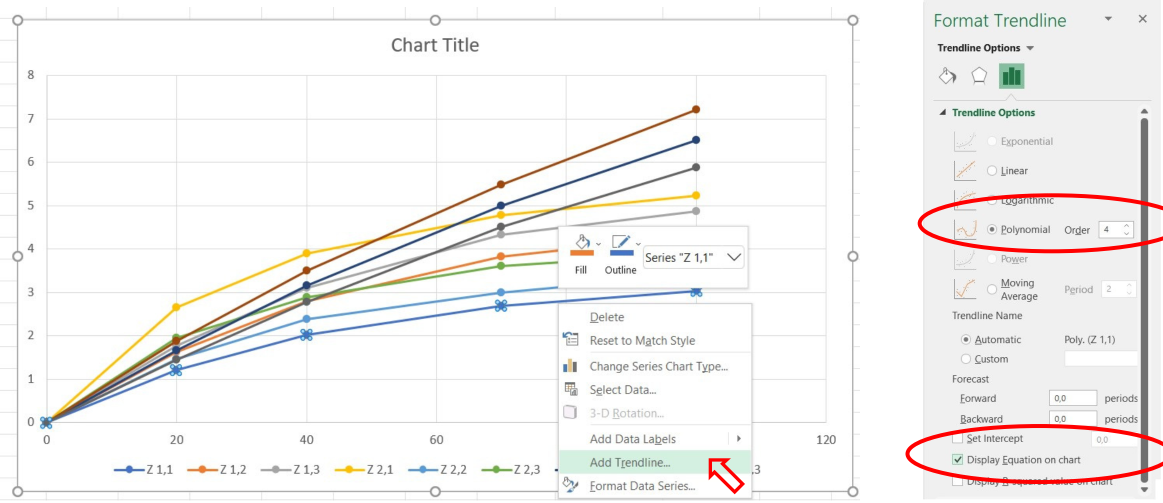

Fig. iv

In the pane Format Trendline, a Polynomial is selected and an order 3 or 4 is set. Also, option Display Equation on chart is selected.

Thus, it is assumed to calculate the costs *Z* by the polynomial: $Z = aV + bV^2 + cV^3 + dV^4$ in which the variable *V* is the supplied volume *V*. It is evident that all functions cross the origin of coordinate system, thus a constant (free term) of the polynomial function is equal to zero and may not be written.

Then a new table, as shown on the right of the **Fig. v** is made. In that table all of the polynomial coefficients are written.

COSTS READ FROM THE GRAPH					
	Volumes of Water				
Cost	0	20	40	70	100
Z 1,1	0	1,21	2,02	2,69	3,03
Z 1,2	0	1,63	2,8	3,82	4,31
Z 1,3	0	1,77	3,11	4,33	4,86
Z 2,1	0	2,65	3,89	4,78	5,23
Z 2,2	0	1,46	2,39	3	3,41
Z 2,3	0	1,95	2,89	3,6	3,89
Z 3,1	0	1,66	3,15	5	6,5
Z 3,2	0	1,88	3,49	5,48	7,21
Z 3,3	0	1,44	2,78	4,5	5,88
Z 4,1					
Z 4,2					
Z 4,3					

	Polynomial terms: $Z = a.V+b.V^2+c.V^3+d.V^4$			
Cost	a	b	c	d
Z 1,1	0,0725	-0,0006	0,00000178	
Z 1,2	0,0956	-0,0007	0,00000175	
Z 1,3	0,0985	-0,0005	-0,000002	2E-08
Z 2,1	0,1866	-0,0033	0,00003	-1E-07
Z 2,2	0,0921	-0,001	0,0000042	
Z 2,3	0,1206	-0,0014	0,000006	
Z 3,1	0,0886	-0,0003	0,00000065	
Z 3,2	0,1027	-0,0004	0,0000009	
Z 3,3	0,0751	-0,0001	-0,0000008	
Z 4,1	1	0	0	
Z 4,2	1	0	0	
Z 4,3	1	0	0	

Fig. v

A new table is made in order to check how good is the match between the function *Z* calculated by means of the obtained equations and the function *Z* on Fig. 2. (see **Fig. vi –** the rightmost table). If the it is needed a polynomial order is decreased to 3 and new values of polynomial coefficients are obtained. As it is seen from the tables in the right side of **Fig. vi**, the cost values read and the cost values calculated are pretty close, thus two functions match good enough.

AE14 fx =AE$13*$R14+AE13^2*S14+AE13^3*T14+AE13^4*U14

	Polynomial terms: $Z = a.V+b.V^2+c.V^3+d.V^4$			
Cost	a	b	c	d
Z 1,1	0,0725	-0,0006	0,00000178	
Z 1,2	0,0956	-0,0007	0,00000175	
Z 1,3	0,0985	-0,0005	-0,000002	2E-08
Z 2,1	0,1866	-0,0033	0,00003	-1E-07
Z 2,2	0,0921	-0,001	0,0000042	
Z 2,3	0,1206	-0,0014	0,000006	
Z 3,1	0,0886	-0,0003	0,00000065	
Z 3,2	0,1027	-0,0004	0,0000009	
Z 3,3	0,0751	-0,0001	-0,0000008	
Z 4,1	1	0	0	
Z 4,2	1	0	0	
Z 4,3	1	0	0	

	COSTS READ FROM THE GRAPH					CALCULATED COSTS BY TRENDLINE FUNCTIONS				
	Volumes of Water					Volumes of Water				
Function	0	20	40	70	100	0	20	40	70	100
Z 1,1	0	1,21	2,02	2,69	3,03	0	1,224	2,054	2,746	3,030
Z 1,2	0	1,63	2,8	3,82	4,31	0	1,646	2,816	3,862	4,310
Z 1,3	0	1,77	3,11	4,33	4,86	0	1,757	3,063	4,239	4,850
Z 2,1	0	2,65	3,89	4,78	5,23	0	2,636	3,848	4,781	5,660
Z 2,2	0	1,46	2,39	3	3,41	0	1,476	2,353	2,988	3,410
Z 2,3	0	1,95	2,89	3,6	3,89	0	1,900	2,968	3,640	4,060
Z 3,1	0	1,66	3,15	5	6,5	0	1,657	3,106	4,955	6,510
Z 3,2	0	1,88	3,49	5,48	7,21	0	1,901	3,526	5,538	7,170
Z 3,3	0	1,44	2,78	4,5	5,88	0	1,456	2,793	4,493	5,710
Z 4,1						0	20	40	70	100
Z 4,2						0	20	40	70	100
Z 4,3						0	20	40	70	100

Fig. vi.

The Cost functions are obtained by means of MS Excel.

At first, for each Cost function $Z_{i,j}$ they are made 4 reading from graphs, shown on Fig. 2. Then chart is inserted in MS Excel using the data from readings. For each function $Z_{i,j}$ a trendline is added and the trendline equation is found. All the Cost functions are described as polynomes. Then a check is performed in order to verify the trendline equations. The results are presented on Fig. 3.

The Cost functions for the virtual water source – Z_{41}, Z_{42} и Z_{43}, are presented as linear functions, which return very high costs – 15 to 20 times bigger, compared to the costs for delivery from real water sources to water users.

AE14 =AE$13*$R14+AE13^2*S14+AE13^3*T14+AE13^4*U14

	Polynomial terms: Z = a.V+b.V²+c.V³+d.V⁴			
Cost	a	b	c	d
Z 1,1	0,0725	-0,0006	0,00000178	
Z 1,2	0,0956	-0,0007	0,00000175	
Z 1,3	0,0985	-0,0005	-0,000002	2E-08
Z 2,1	0,1866	-0,0033	0,00003	-1E-07
Z 2,2	0,0921	-0,001	0,0000042	
Z 2,3	0,1206	-0,0014	0,000006	
Z 3,1	0,0886	-0,0003	0,00000065	
Z 3,2	0,1027	-0,0004	0,0000009	
Z 3,3	0,0751	-0,0001	-0,0000008	
Z 4,1	1	0	0	
Z 4,2	1	0	0	
Z 4,3	1	0	0	

	COSTS READ FROM THE GRAPH					CALCULATED COSTS BY TRENDLINE FUNCTIONS				
	Volumes of Water					Volumes of Water				
Function	0	20	40	70	100	0	20	40	70	100
Z 1,1	0	1,21	2,02	2,69	3,03	0	1,224	2,054	2,746	3,030
Z 1,2	0	1,63	2,8	3,82	4,31	0	1,646	2,816	3,862	4,310
Z 1,3	0	1,77	3,11	4,33	4,86	0	1,757	3,063	4,239	4,850
Z 2,1	0	2,65	3,89	4,78	5,23	0	2,636	3,848	4,781	5,660
Z 2,2	0	1,46	2,39	3	3,41	0	1,476	2,353	2,988	3,410
Z 2,3	0	1,95	2,89	3,6	3,89	0	1,900	2,968	3,640	4,060
Z 3,1	0	1,66	3,15	5	6,5	0	1,657	3,106	4,955	6,510
Z 3,2	0	1,88	3,49	5,48	7,21	0	1,901	3,526	5,538	7,170
Z 3,3	0	1,44	2,78	4,5	5,88	0	1,456	2,793	4,493	5,710
Z 4,1						0	20	40	70	100
Z 4,2						0	20	40	70	100
Z 4,3						0	20	40	70	100

Fig. 3.

2.3.2. MS Excel model

Two matrices are made – the Matrix of volumes and Matrix of Costs (see Fig. 4).

F4 =SUM(C4:E4)

MATRIX OF VOLUMES

	B1	B2	B3	Sum	Available V
A1	20	0	0	20	40
A2	0	0	0	0	40
A3	0	0	0	0	20
A4	0	0	0	0	20
Sum	20	0	0		120
Needed V	30	60	30	120	

MATRIX OF COSTS

	B1	B2	B3	Sum	
A1	1,22424	0	0	1,224	
A2	0	0	0	0,000	
A3	0	0	0	0,000	
A4	0	0	0	0,000	
				1,224	
Sum	1,22424	0	0	1,224	Check

TYPE =$R$14*C4+$S$14*C4^2+$T$14*C4^3+$U$14*C4^4

MATRIX OF VOLUMES

	B1	B2	B3	Sum	Available V
A1	20	0	0	20	40
A2	0	0	0	0	40
A3	0	0	0	0	20
A4	0	0	0	0	20
Sum	20	0	0		120
Needed V	30	60	30	120	

MATRIX OF COSTS

	B1	B2	B3	Sum	
A1	*C4+$S$1	0	0	1,224	
A2	0	0	0	0,000	
A3	0	0	0	0,000	
A4	0	0	0	0,000	
				1,224	
Sum	1,22424	0	0	1,224	Check

Fig. 4.

The Matrix of volumes. In the rows of Matrix of volumes, the number of water sources are set, and in the columns – the number of water users. Thus, in each cell the volume from source A_i to user B_j will be written. In the column Sum, the sum of the supplied volumes from source A_i is calculated. In the column Available V (cells in yellow), the capacity W_i^A of each water source is entered (see Fig. 4 – left picture).

In the row Sum, the sum of the supplied volumes to water user source B_j is calculated. In the row Needed V (cells in yellow), the demand W_j^B of each water user is entered.

In the **Matrix of costs**, in each cell it is calculated the cost for supplying water from source A_i to user B_j in dependence of the volume supplied from source A_i to user B_j and the Cost function for that route. As it is seen from Fig. 4 – right picture, the value of the volume is taken from the Matrix of volumes and the polynomial coefficients – from the left table on Fig. 3.

The column Sum contains the sum of costs for supplying water from source A_i to all users B. The row Sum contains the sum of costs for supplying water from all sources A to user B_j. The value in the cell F18 contains the sum of the values above, and the value in the cell F19 contains the sum of the values from the left. Both sums in cells F18 and F19 have to be the same. The value of the cell F18 (and in cell F19) is the value of Objective Function.

An initial solution, so called ***basic feasible solution*** should be done for the volumes in the Matrix of volumes.

One feasible solution is shown on **Fig. vii**.

F19 =SUM(C19:E19)

MATRIX OF VOLUMES

	B1	B2	B3	Sum	Available V
A1	30	10	0	40	40
A2	0	40	0	40	40
A3	0	10	10	20	20
A4	0	0	20	20	20
Sum	30	60	30		120
Needed V	30	60	30	120	

MATRIX OF COSTS

	B1	B2	B3	Sum	
A1	1,68306	0,88775	0	2,571	
A2	0	2,3528	0	2,353	
A3	0	0,9879	0,7402	1,728	
A4	0	0	20	20,000	
				26,652	
Sum	1,68306	4,22845	20,7402	26,652	Check

The basic feasible solution can be done using so-called Northwest Corner Method.

In this method we start from the top leftmost cell. In that cell we write as big as possible volume, according to capacity or the demand for the first water source and the first water user. Then, we move to adjacent cell – either right, or down, depending which one of the constraints (capacity or demand) is not fulfilled.

Thus, moving generally from Northwest corner to Southeast corner we allocate in each cell as big volume is possible.

Fig. vii

The task is solved by means of Excel Solver. This is a built-in add-on tool in MS Excel. If it is activated it can be found under the Data tab. If it is not activated, then go to File → Options → Add-ins. Select Solver Add-in → Manage → Solver → OK. Then you can find it under Data tab, on the right side of the toolbar, in section Analyze.

Select cell F18 and then start Solver.

The dialog box appears (see **Fig. viii**).

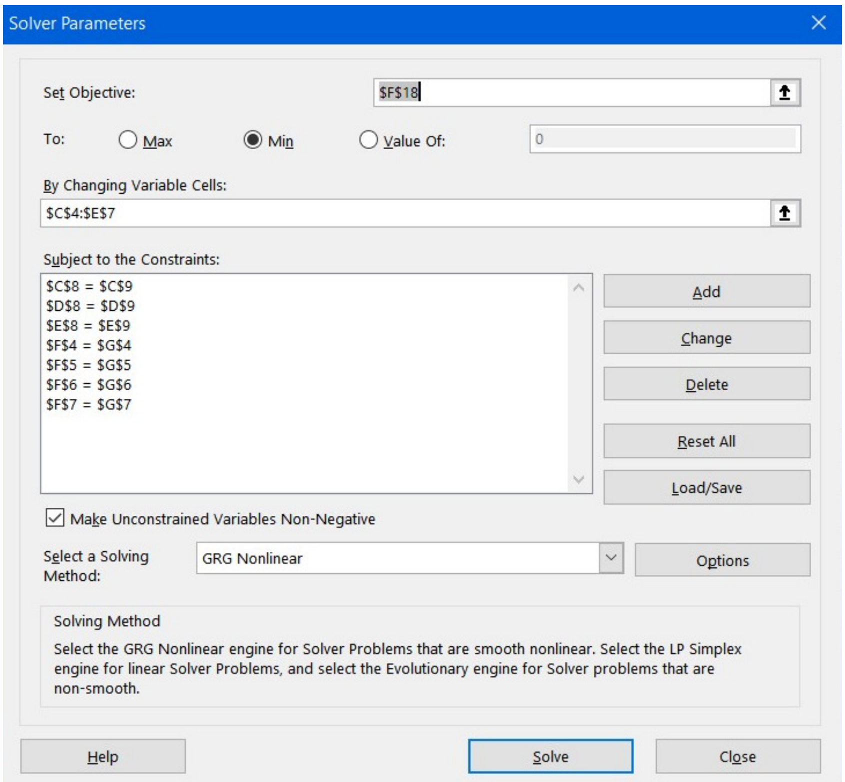

Fig. viii

It is automatically suggested by the Solver that the OF is in marked cell. In this case it is cell F18. The formula in cell F18 calculates the value of the OF, as it is defined by equation (2).

We look for *minimum* value of the Objective Function, thus we select radio button Min.

In the box By Changing Variable Cells we point (select) the range of the volumes in the Matrix of volumes. For this example, this is the range from cell C4 to cell E7. when we select the range, automatically the absolute addresses of cells appear.

In the box Subject to the Constraints we have to specify the constraints. They are described by the set of equations (3) and (4). It should be inserted, once at the time, each constraint. As it is seen from **Fig. vii** and **Fig. viii** (the first 3 constraints) the sum of the supplied volumes to each water user B_j should be equal to its demand – these represent constraints in equation (4). The same is valid for the water sources (the last 4 constraints)– the sum of the supplied volumes from each water source A_i should be equal to its capacity – these represent constraints in equation (3).

Since all the variables (i.e. volumes) should not be negative, we select the check box Make Unconstrained Variables Non-Negative.

If it is not specified or some other method is suggested, we have to select from the drop-down menu Select a Solving Method the option GRG Nonlinear.

Then we click on the button Solve.

When a feasible solution is found the notification pops up. The solution is ready and we click on Keep Solver Solution.

CAUTION! It is advisable to try several Solver solutions, each one of them using different *basic feasible solution*. Sometimes the results preformed by Solver are not correct due to nonlinear relations.

On **Fig. ix** there is an example of one initial feasible solution, and a final solution, which is declared *optimal* by Solver. However, as it is seen from Fig. 5, this is actually NOT the optimal solution.

	A	B	C	D	E	F	G
1							
2		MATRIX OF VOLUMES					
3			B1	B2	B3	Sum	Available V
4		A1	20	10	10	40	40
5		A2	10	10	20	40	40
6		A3	0	20	0	20	20
7		A4	0	20	0	20	20
8		Sum	30	60	30		120
9		Needed V	30	60	30	120	
10							
11							
12		MATRIX OF COSTS					
13			B1	B2	B3	Sum	
14		A1	1,22424	0,88775	0,9332	3,045	
15		A2	1,565	0,8252	1,9	4,290	
16		A3	0	1,9012	0	1,901	
17		A4	0	20	0	20,000	
18						**29,237**	
19		Sum	2,78924	23,6142	2,8332	29,237	Check

	A	B	C	D	E	F	G
1							
2		MATRIX OF VOLUMES					
3			B1	B2	B3	Sum	Available V
4		A1	30	10	0	40	40
5		A2	0	10	30	40	40
6		A3	0	20	0	20	20
7		A4	0	20	0	20	20
8		Sum	30	60	30		120
9		Needed V	30	60	30	120	
10							
11							
12		MATRIX OF COSTS					
13			B1	B2	B3	Sum	
14		A1	1,68306	0,88775	0	2,571	
15		A2	0	0,8252	2,52	3,345	
16		A3	0	1,9012	0	1,901	
17		A4	0	20	0	20,000	
18						**27,817**	
19		Sum	1,68306	23,6142	2,52	27,817	Check

Fig. ix

2.3.3. MS Excel Solver Solution

The solution of the task is obtained by means of the Solver add-in tool of MS Excel.

The results are presented in Fig. 5.

MATRIX OF VOLUMES

	B1	B2	B3	Sum	Available V
A1	30	10	0	40	40
A2	0	40	0	40	40
A3	0	0	20	20	20
A4	0	10	10	20	20
Sum	30	60	30		120
Needed V	30	60	30	120	

MATRIX OF COSTS

	B1	B2	B3	Sum	
A1	1,68306	0,88775	0	2,571	
A2	0	2,3528	0	2,353	
A3	0	0	1,4556	1,456	
A4	0	10	10	20,000	
				26,379	
Sum	1,68306	13,2405	11,4556	26,379	Check

Fig. 5. Problem solution by means of MS Excel Solver

The real minimum of the Objective function is obtained by reducing the costs for delivery from dummy water source to water users. In this case the costs are 20 thousand Euro.

The real costs for water delivery are:

$Cost$ = 26,379 – 20,000 = 6,379 thousand Euro or 6 379 €.

It can be seen from the table on Fig. 5, that the demand of the water users B_2 and B_3 is not satisfied. It was evident that the system has a deficit of 20 000 m^3 and the most economical solution is to have deficits of 10 000 m^3 for each of water users B_2 and B_3. This may not be satisfying solution for user B_3, because the deficit represents 33% of its demand.